Numerical Proof of Self-Similarity in Burgers' Turbulence

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Abstract

We study the statistical properties of solutions to Burgers' equation, $v_t + vv_x = \nu v_{xx}$, for large times, when the initial velocity and its potential are stationary Gaussian processes. The initial power spectral density at small wave numbers follows a steep power-law $E_0(k) \sim |k|^n$ where the exponent n is greater than two. We compare results of numerical simulations with dimensional predictions, and with asymptotic analytical theory. The theory predicts self-similarity of statistical characteristics of the turbulence, and also leads to a logarithmic correction to the law of energy decay in comparison with dimensional analysis. We confirm numerically the existence of self-similarity for the power spectral density, and the existence of a logarithmic correction to the dimensional predictions.

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1 Introduction

We study the partial differential equation

$$v_t + vv_x = \nu v_{xx} \tag{1}$$

in the limit of vanishing coefficient ν. Here $v(x,t)$ can be taken as the velocity of a wave field, a function of one-dimensional spatial coordinate x and time t, and ν as the viscosity of the medium.

This equation, first introduced by J.M. Burgers [1] as a model of hydrodynamic turbulence, arises in many situations in physics. In the theory of wave propagation Burgers' equation (1) describes the evolution of nonlinear waves in non-dispersive dissipative media [22]. An important example of such waves is nonlinear acoustic plane waves [14]. When a random force is added to the right-hand side it describes surface growth phenomena [10, 16]. Burgers' equation is an effective equation on large scales, when parity-invariance holds, and arises as an approximation in description of large-scale structures in the Landau-Ginsburg equation [11]. One further non-trivial application is to model the formation and evolution of large-scale structures in the Early Universe, within the so-called adhesion approximation [5, 7, 17].

From the equation (1) the following equation for the potential of the velocity,

$$v = -\psi_x, \tag{2}$$

is derived:

$$\psi_t = \frac{1}{2}(\psi_x)^2 + \nu\psi_{xx}. \tag{3}$$

By using the Hopf-Cole transformation, $\psi = 2\nu \ln\theta$, equation (2) is transformed to the linear diffusion equation [9, 2]:

$$\theta_t = \nu\theta_{xx}. \tag{4}$$

The solution of (4) is found by standard convolution with the heat kernel, which in the limit $\nu \to 0$ may be evaluated by the method of steepest descent.

The solution of Burgers' equation (1) in the limit $\nu \to 0$ then has the form:

$$v(x,t) = \frac{x - a(x,t)}{t} \tag{5}$$

where $a(x,t)$ is the coordinate where the maximum of the function

$$G(x,a,t) = \psi_0(a) - \frac{(x-a)^2}{2t}. \tag{6}$$

is obtained. The velocity potential is found as

$$\psi(x,t) = \max_a \left[G(x,a,t) \right]. \tag{7}$$

In the continuum there are two equivalent ways of computing $v(x,t)$. Either, the velocity may be found by taking the derivative of the function $\psi(x,t)$ from (7), using that the velocity is the gradient of the velocity potential (2), or equation (5) may be used directly. In a discrete realization the two ways are still equivalent, provided we make the correct interpolation between the grid points. For long times the difference is not important, and will be ignored in the following.

We use the following definition of the spectral density of energy:

$$E(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} < v(x+s,t)v(x,t) > \mathrm{e}^{iks}\, ds, \tag{8}$$

and we are interested in the case where the initial data is a Gaussian process, with continuous spectral support around a dominant wave-number k_0. We want to know how the spectrum changes with time, for sufficiently long times. Our approach applies when the initial energy spectrum grows faster than k^2 when k is much less than k_0, and decays quickly when k is larger than k_0. More to the point, we consider the case where the initial spectrum is a power-law at small k

$$E_0(k) \sim |k|^n \qquad |k| << k_0 \tag{9}$$

where the exponent n is a real number greater than two. At large k we have a smooth cut-off. The construction of the initial conditions including normalization factors is described below in section 5.1.

2 Summary of dimensional predictions

We assume a dominant initial wave band around k_0, which implies a correlation length of the initial data $l_0 \sim 1/k_0$.

From the characteristic length and the fluctuations of initial velocity we can form a characteristic time t_{nl}, the nonlinear time or the characteristic time of shock formation:

$$t_{nl} = \frac{l_0}{\sqrt{< v_0^2 >}}. \tag{10}$$

As an illustration we show in fig. 1 one typical realization of the initial conditions, and how it develops in time. The characteristic scale l_0 can be estimated as the average distance between local maxima (or minima) of the smooth initial data (fig. 1a). At the nonlinear time the initial data has changed to a saw-tooth wave with approximatively the same characteristic scale (fig. 1b). At larger times the velocity gradient between the shocks decreases, and the characteristic scale increases (fig. 1c). This behaviour is intimately related to the change of the spectral shape with time, to which we now turn.

We will consider in the following only the asymptotic solutions of (1) at times much greater than the nonlinear time,

$$t \gg t_{nl}. \tag{11}$$

The characteristic length scale at this time is derived by balancing in order of magnitude the two terms in the Hopf-Cole condition (6,7):

$$\sqrt{< \psi_0^2 >} \sim \frac{< (x - a(x))^2 >}{t} \sim \frac{l^2}{t}. \tag{12}$$

The distances over which the maximizing is effectively taken at time t are therefore such that

$$l(t) \sim \left(t\sqrt{< \psi_0^2 >} \right)^{\frac{1}{2}}, \tag{13}$$

or, in more transparent form,

$$l(t) \sim l_0 (t/t_{nl})^{\frac{1}{2}}. \tag{14}$$

We therefore have the picture that at time t the solutions are composed of shocks, separated by a typical distance $l(t)$. The spatial gradient of velocity between shocks is $1/t$ from (5). The typical velocity and the typical amplitude of velocity jumps across shocks is therefore $l(t)/t$. From these considerations, the mean energy in the velocity field per unit length at time t is:

$$E(t) =< v^2 > \sim \frac{l(t)^2}{t^2} \sim < v_0^2 > (t/t_{nl})^{-1}. \tag{15}$$

Qualitatively, it is possible to estimate the behavior of the power spectral density from the following considerations. As is well known, the development of shocks leads to the asymptotic of the spectrum at large wave-numbers, $E(k,t) \sim k^{-2}$, and the parametric pumping of energy to the area at small k's leads to the universal quadratic law, $E(k,t) \sim k^2$ [1, 5]. Therefore the spectrum should consist of two branches, smoothly transformating from one asymptotics k^2 to another k^{-2}. At the turnover point, k^*, the spectrum has a maximum:

$$E(k,t) \sim \begin{cases} A(t)k^2, & \text{if } k \ll k^* \\ B(t)k^{-2}, & \text{if } k \gg k^* \end{cases} . \tag{16}$$

The only possible scale for k^* is

$$k^*(t) \sim \frac{1}{l(t)}. \tag{17}$$

For the prefactors in (16), one obtains by comparing with the total energy (15) and matching at k^*:

$$A(t) \sim < v_0^2 > l_0^3 \left(\frac{t}{t_{nl}}\right)^{1/2}, \tag{18}$$

$$B(t) \sim \frac{< v_0^2 >}{l_0} \left(\frac{t}{t_{nl}}\right)^{-3/2}. \tag{19}$$

We see that (18) and (19) could also have been derived directly by dimensional analysis, since $A(t) \sim \frac{l^5(t)}{t^2}$ and $B(t) \sim \frac{l(t)}{t^2}$.

It should be noted that if ν is small but finite we can form a ratio between the dissipation time over spatial scale l_0 and the nonlinear time, which we can naturally refer to as the initial Reynolds number.

$$Re_0 = \frac{l_0^2/\nu}{t_{nl}} = \frac{l_0 < v_0^2 >^{1/2}}{\nu} \tag{20}$$

Similarly, at time t we can form the ratio between the dissipation time over spatial scale $l(t)$ and the turn-over time, which is on the order of $l(t)/\sqrt{< v^2(t) >}$, that is, simply proportional to t:

$$Re(t) = \frac{l^2(t)/\nu}{t} \sim \frac{l_0 < v_0^2 >^{1/2}}{\nu} \tag{21}$$

Within dimensional considerations the Reynolds number hence remains constant, because the increase in $l(t)$ cancels with the decrease in typical velocity.

Another way to introduce the Reynolds number, is to observe that the width of a shock is of the order of $\nu/\sqrt{< v^2(t) >}$. Reynolds number is thus also equal to the ratio between the inter-shock distance at time t, that is $l(t)$, and the width of the shock. Therefore, as long as Reynolds number stays the same, the Hopf-Cole transformation and approximation that the solutions have saw-tooth shape are relatively equally accurate at arbitrarily large times.

We will see in the next section that by including logarithmic corrections to the dimensional estimates, Reynolds number will in fact slowly decrease in time. There will be a characteristic time over which Reynolds number decreases to order unity, and then viscosity can not be taken small. This time will however be exponentially large in the square of Re_0, so if viscosity is initially weak it can be taken weak for very long time, although not infinitely long.

3 Beyond the dimensional predictions

For Burgers' turbulence, in the cases that interest us here, it is possible to develop an asymptotic theory going beyond dimensional analysis, which is valid at times much greater than t_{nl}. From the mathematical point of view we use the property that for large times the parabola in (6) varies little over a distance of the order of $l(t)$ at its minimum. When $l(t)$ is much larger than l_0, and the correlations in the initial conditions decay sufficiently fast beyond l_0, then a large number of local maxima of the initial potential $\psi_0(a)$ competes to be the global maximum of the function $G(x, a, t)$, and these local minima are practically independent.

We can then appeal to result in probability theory on the expected maxima of Gaussian processes [3, 20]. A more direct, although more cumbersome, approach to derive the asymptotics also exists[4], which has in addition the advantage that one may thus also compute order by order corrections to the leading behaviour[8].

We begin by observing that the mean square velocity and mean square velocity potential are both finite for the process we consider. By comparing

them we can thus form a characteristic length

$$l_0 = \sqrt{\frac{<\psi_0^2>}{<v_0^2>}}, \tag{22}$$

which will of course also be of the order $1/k_0$, but with a well-defined finite prefactor depending on the functional form of the spectrum.

Let us consider the decay of energy with time. Averaging both sides (3) over space leads to

$$\partial_t <\psi> = E(t) \tag{23}$$

The decay of energy can thus be derived from the expectation value of the velocity potential at time t. The maximization position (a) in (7) can at most be a distance of leading order $\sqrt{t}$ away from the Eulerian coordinate (x). The expectation value of the velocity potential at time t should therefore be similar to the expected maximum value of the initial velocity potential on a stretch of the line of length $\sqrt{t}$. One would expect this value to grow in time, although not very fast. In fact, a well-known result on stationary Gaussian processes[3, 20, 5] states that this maximum has a doubly exponential distribution, such that the expected maximum on a stretch of the line of length L is $<\psi_0^2>^{1/2}\sqrt{2\log\frac{L}{l_0\sqrt{2\pi}}}$, where $<\psi_0^2>$ has the same meaning as introduced above, and l_0 as in equation (22).

Substituting $l_0(t/t_{nl})^{1/2}$ for L we arrive at

$$<\psi(t)> \sim \sqrt{<\psi_0^2>}\sqrt{\log\frac{t}{2\pi t_{nl}}} \tag{24}$$

Although we should not a priori expect that (24) is correct up to constants, it turns out to be the case [12, 4, 6, 8]. By differentiating in t follows Kida's law for the the deacy of energy at long times:

$$E(t) = \frac{1}{2} <v_0^2> (t/t_{nl})^{-1} \left(\log(t/2\pi t_{nl})\right)^{-1/2} \tag{25}$$

Again, this expression holds in the asymptotic theory up to additive corrections that become small relative to (25) in the limit.

We can also estimate the behaviour of the characteristic length. When time grows, the expected maximum of the term ψ_0 in (7) will slowly become larger than $(x-a)^2/2t$, if the effective length of the maximization operation,

$(x-a)$, is just taken to be of order $\sqrt{t}$. The balance equation, (12), therefore slowly becomes inappropriate to fix the spatial scale. Let us therefore instead assume that the maximum is attained at a value of $(x-a)$ around L. The attained value of $\psi(x,t)$ would then typically be

$$< \psi >_L \sim \left(\sqrt{< \psi_0^2 >} \sqrt{2 \log \frac{L}{l_0 \sqrt{2\pi}}} \right) - \frac{L^2}{2t}. \tag{26}$$

If we now maximize (26) over L we find that the maximum is attained at a value of

$$l(t) \sim l_0 (t/t_{nl})^{1/2} (\log \frac{t}{2\pi t_{nl}})^{-\frac{1}{4}} \tag{27}$$

The characteristic length-scale therefore grows slightly more slowly than the dimensional estimate. Let us note that the estimate of the maximum (24) remains unchanged. Only subleading terms are influenced by the small correction to the characteristic length. Let us also note that the dimensional estimate $E(t) \sim l^2(t)/t^2$ remains correct up to logarithmic corrections, which indicates that the fluctuations are very small.

In a similar spirit one may also compute two-point probability densities [5, 6], correlation functions and the power spectral density [12, 4, 6, 5]. In particular, the energy spectrum has the expression

$$E(k,t) = \frac{l^3(t)}{t^2} \tilde{E}(kl(t)), \tag{28}$$

where $l(t)$ is the characteristic scale in (27), and the function $\tilde{E}(\kappa)$ can be computed to the same leading order in the asymptotics[5, 6, 12].

In fig. 2 (a) three power spectra (averaged over 1000 realizations of the random process) are shown at three different moments of time. The initial spectrum was k^3 at small k. The preservation of the shape of each curve is evident. It is also seen that the total energy (the area under the curves) decreases, and so does the characteristic wave-number k^*. Fig. 2 (b) contains three power spectra computed at the same time at the self-similar stage (averaged over 1000 realizations). The initial spectra were k^3, k^6 and k^{12} at small k, respectively.

The analytical expression for $\tilde{E}$ is somewhat unwieldy. However, qualitatively, $\tilde{E}$ is simply described as a smooth interpolation between two power-law asymptotes that hold when κ is respectively much smaller or much larger

than unity. In the first case $\tilde{E}(\kappa) \sim 1.08\kappa^2$, in agreement with the expected leading k^2 behaviour of the spectrum according to (16), while for large κ $E(\kappa) \sim 0.359/\kappa^2$, also in agreement with (16) [5]. In Fig.3 $\tilde{E}(\kappa)$ is shown together with numerical results obtained by averaging over 1000 realizations of the initial conditions. The numerically obtained curve (as a result of averaging over 1000 realizations of the initial random process) is artificially shifted slightly downwards to be distinguishable from the analytical curve.

It is seen that the asymptotic formula describes the numerical data very well, not only in the limits of relatively large and small wave numbers, but also at the top, where the spectrum switches between the two asymptotes. From fig. 3 is also seen that the transformation region from the top down towards κ^{-2} is thin, but the transformation region from the top down to κ^2 is considerably wider.

In fig. 4 the function $l(t)$ is shown. The dashed line corresponds to the calculation based on the expression (13). The result of the computations taking into account the logarithmic correction (27) is shown by the solid line. The centered symbols show results from numerical simulations of an ensemble of 500 realizations. The length scale was estimated from the numerical data in the following way: the averaged spectrum was first found, and by fitting the two asymptotes Ak^2 and Bk^{-2}, their intersection was taken as the main wave number k^*. The numerical estimate of $l(t)$ was then defined as $1/k^*$. The fact that the results with the logarithmic corrections fit the data better proves numerically the validity of the asymptotic theory.

In fig. 5 we show the results of a procedure analogous to that depicted in fig. 4 but for the function $E(t)$. It is seen again that the solid line obtained by taking into account the logarithmic correction fits the numerics (centered symbols) better. Since the correction is slow, the difference between this result and the result of the dimensional analysis (dashed line) is visible but small.

From which have been shown above one might conclude that the logarithmic correction is not very significant, since it gives a small addition to the dimensional predictions. To a certain extent that depends on what quantity one looks at. We can for instance compute the prefactors of the power-law for the spectrum at large and small wave numbers. If $E(k) \sim A(t)k^2$ up to a maximum wave number k^*, then the total energy at time t is to leading order $A(t)(k^*)^3$. But we know already the energy and k^* as functions of t, so by comparing we compute $A(t)$. By matching at k^* we then also find $B(t)$,

and the results are:

$$A(t) \sim < v_0^2 > l_0^3 \left(\frac{t}{t_{nl}}\right)^{1/2} \left(\log \frac{t}{2\pi t_{nl}}\right)^{-\frac{5}{4}} \tag{29}$$

$$B(t) \sim \frac{< v_0^2 >}{l_0} \left(\frac{t}{t_{nl}}\right)^{-3/2} \left(\log \frac{t}{2\pi t_{nl}}\right)^{-\frac{1}{4}} \tag{30}$$

In the case of $A(t)$, the correction to the dimensional predictions is more significant than those shown previously. In fig. 6 the centered symbols show the values of $A(t)$ obtained from the numerical simulation. The dashed line is the dimensional prediction computed according to (18), while the solid line takes into account the logarithmic correction (29).

There is also a real physical interest in the logarithmic corrections, since they change qualitatively the behaviour at large times. Let us form the Reynolds number at time t, as in (21):

$$Re(t) = \frac{l(t)\sqrt{E(t)}}{\nu} \sim Re_0 \left(\log \frac{t}{2\pi t_{nl}}\right)^{-\frac{1}{2}} \tag{31}$$

where Re_0 is the initial Reynolds number as in (20). Although we compute the solutions to Burgers' equation in the limit where viscocity vanishes, that is, in the infinite Reynolds number limit, Reynolds number still decreases with time [1, 5, 6]. Sooner or later we will reach a stage where viscosity can not be taken arbitrarily small, and at even later time we will eventually reach a the linear stage of decay. The time to reach the linear regime can be estimated by taking $Re(t_l) \sim 1$, which leads to

$$t_l \sim 2\pi t_{nl} \exp(Re_0^2). \tag{32}$$

4 Different dimensional ansätze

We started our more precise analysis by expressing the correlation length as $l_0 = \sqrt{\frac{<\psi_0^2>}{<v_0^2>}}$. We might also consider the initial velocity gradient, and its variance, that we write $\sigma_u^2 =< (\partial_x v_0)^2 >$. We could then define the correlation length by $l_0' = \sqrt{\frac{<v_0^2>}{\sigma_u^2}}$, and a new nonlinear time by $t_{nl}' = 1/\sigma_u$. This definition of the nonlinear time has some appeal, since a shock first

forms where the velocity gradient has a local minimum, and does so after a time $1/|\partial_x v_0|$.

For initial spectra going as k^n with n greater than two, l_0' and t_{nl}' are of course simply proportional to l_0 and t_{nl}. In dimensional analysis it does not matter which ones we use, but, as we have seen, in the asymptotic analysis the correlation length and the nonlinear time that appear are those given above and used in section 3. Furthermore, a considerable part of the asymptotic analysis is valid down to n equal to one [8]. In that limit l_0 diverges, while l_0' remains finite. The expressions from dimensional analysis using l_0' and t_{nl}' therefore grow progressively less accurate as n tends to one from above.

If we lower n further and consider the interval $[-1, 1]$, l_0 and t_{nl} do not exist, because the initial velocity potential is then not a stationary process, only a process with stationary increments. The initial velocity will still be stationary however, and its correlation length will be approximatively l_0', and the shock formation time will still be about t_{nl}'. For distances $|a - x|$ much greater than l_0', $< (\psi_0(a) - \psi_0(x))^2 >$ is equal to $\beta^2(|a - x|)^{1-n}$, where β^2 is given as $\alpha^2 \frac{4\pi}{\Gamma(2-n)\cos\frac{\pi n}{2}}$. Balancing in (7) gives $l(t) \sim (\beta t)^{\frac{2}{3+n}}$. We see that β^2 diverges as n tends to one from below.

If we instead proceed by dimensional analysis using l_0', then $< (\psi_0(a) - \psi_0(x))^2 >$ should be similar to $(l_0')^2 < v_0^2 > (\frac{|a-x|}{l_0'})^{1-n}$, with a numerical factor which depends on how the cut-off is made at large k. Balancing in (7) gives $l(t) \sim l_0' \, (t/t_{nl}')^{\frac{2}{3+n}}$. The upshot of this discussion is that t_{nl}' is a good measure of shock formation time, and can be used as a nonlinear time in the dimensional estimates in the whole interval of n greater than -1, except around one, where numerical prefactors diverge, because the long-term asymptotics changes qualitatively around this point.

5 Numerical work

5.1 Normalizations of initial spectrum

We use the following smooth cut-off of the initial power spectrum:

$$E_0(k) = \alpha_n^2 |k|^n \mathrm{e}^{-\frac{k^2}{2k_0^2}}. \tag{33}$$

We let k vary in the first Brillouin zone, that is k goes between $-1/2$ and $1/2$, with k_0 of the order of $1/2$. The initial spatial scale l_0 is therefore of order one.

The variances of the velocity and the velocity potential can be computed to be:

$$< v_0^2 > = \alpha_n^2 \Gamma\left(\frac{n+1}{2}\right) k_0^{n+1}, \tag{34}$$

$$< \psi_0^2 > = \alpha_n^2 \Gamma\left(\frac{n-1}{2}\right) k_0^{n-1}, \tag{35}$$

where $\Gamma(x)$ is the gamma-function.

Furthermore, we use the convention that $< \psi_0^2 >$ is equal to one, which is of course equivalent to making a choice of scale of time. We have then

$$< v_0^2 > = \frac{\Gamma\left(\frac{n+1}{2}\right) k_0^2}{\Gamma\left(\frac{n-1}{2}\right)} \tag{36}$$

and, following the above,

$$l_0 = \frac{1}{k_0}\sqrt{\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}} \tag{37}$$

and

$$t_{nl} = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right) k_0^2}. \tag{38}$$

We can also compute the variance of the velocity gradient, which is

$$\sigma_u^2 = \alpha_n^2 \Gamma\left(\frac{n+3}{2}\right) k_0^{n+3}, \tag{39}$$

We can hence form the second estimate of the initial correlation lengths, l_0', and the ratio between the two lengths, which is

$$\frac{l_0'}{l_0} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+3}{2}\right)}}, \tag{40}$$

We see indeed that these two lengths only differ by a constant depending on n, as was stated in the previous discussion.

5.2 Generation of initial conditions

Fourier components of a Gaussian process are independent Gaussian variables. We therefore synthesize the initial potential of the velocity by first generating random variables a_k distributed according to

$$P(a_k) = \frac{1}{N} \exp\left(-\frac{a_k^2}{2\sigma_k^2}\right)$$

where

$$\sigma_k^2 = E_{\psi_0}(k)dk.$$

Here is used the well known relation between the power spectra of the process and its derivative

$$E_{\psi_0}(k) = k^{-2} E_0(k) \tag{41}$$

where the form of $E_0(k)$ is chosen with a smooth cut-off at large k according to (33). By an inverse Fourier transform we find the initial potential in real space, and repeating the whole process many times we sample the desired ensemble of Gaussian initial conditions.

5.3 Fast Legendre Transforms

In numerical simulations the initial data are always generated as a discrete set of N points. It could be assumed naively that the number of operations necessary to compute the maximization (7) for all values of x scales as $O(N^2)$. It may however be shown, using (7) that $a(x)$ is a nondecreasing function of x. The number of operations needed in an ordered search therefore scales as $O(N \log_2 N)$ when using the so-called Fast Legendre Transform procedure [18, 13].

When the number of grid points is large they can not all fit into the working memory of the CPU at the same time. Even if the operations count is the same, it is preferable to limit paging of data from an external memory storage to a minimum. This is achieved in the in-order algorithm of Noullez [13] that we use here.

5.4 Discrete Fourier analysis

After Legendre transform, we have at our disposal a function $\psi(x,t)$ sampled at N discrete points $\psi(x_l, t)$, $l = 1, 2, ..., N$. The sample interval is $\Delta x = 1$.

Computing the discrete Fourier transform by using FFTs as

$$\hat{\psi}(j,t) = \sum_l \psi(x_l,t) e^{2\pi i j l/N} \quad j = 0,\ldots,N-1 \tag{42}$$

we will get Fourier coefficients at wave numbers $k_j = \frac{\Delta x j}{N}$, where the last half indicates negative frequencies. Since the signal is real the Fourier amplitudes are related by $\hat{\psi}(k_j,t) = \hat{\psi}^*(-k_j,t)$.

Knowing the spectral density of the potential $\hat{\psi}(k_j,t)$, it is easy to find the spectral density of the velocity, since

$$\hat{v}(k_j,t) = ik_j\hat{\psi}(k_j,t).$$

Different definition can then be made of the numerically obtained power spectrum, which only differ by prefactors of N. We use here the normalizations given in a standard reference[15]:

$$\begin{aligned} E(0,t) &= \frac{1}{N^2}|\hat{v}(0,t)|^2, \\ E(k_j,t) &= \frac{2}{N^2}|\hat{v}(k_j,t)|^2,\ j = 1,2,...,\left(\frac{N}{2}-1\right), \\ E(k_c,t) &= \frac{1}{N^2}|\hat{v}(k_c,t)|^2. \end{aligned} \tag{43}$$

Since the initial potential $\psi_0(x_l)$ is obtained by an inverse Fourier transform it is a periodic function with period N. But the use of the fast Legendre transform of a periodic function introduces undesirable edge effects which do not allow to achieve periodicity in the resulted potential $\psi(x_l,t)$. This causes distortion of the spectrum connected with the finite, but not periodic, function in real space. We eliminate this effect by constructing the periodic potential at time t for the cost of using the fast Legendre transform over additional N points of the initial potential. The 'new' periodic initial potential in obtained by simple addition a fragment of $\psi_0(x_l)$ with l in $[N/2+1,N]$ to the left-hand side of the same realization. A fragment with l in $[1,N/2]$ is then added to the right-hand part. Doing this we again obtain a periodic function $\psi_0'(x_l)$ but now with 'double' periodicity

$$\begin{aligned} \psi_0'(x_l) &= \psi_0(x_{l+N/2}), & 1 \le l \le N/2, \\ \psi_0'(x_l) &= \psi_0(x_{l-N/2}), & N/2+1 \le l \le N+N/2, \\ \psi_0'(x_l) &= \psi_0(x_{l-N-N/2}), & N+N/2+1 \le l \le 2N. \end{aligned}$$

After performing the fast Legendre transform over $2N$ points we get a function $\psi'(x_l, t)$. This function loses the 'double' periodicity, but the middle N points of this realization

$$\psi(x_l, t) = \psi'(x_{l+N/2}), 1 \leq l \leq N$$

form a periodic function $\psi(x_l, t)$ with period N which we use further for the Fourier analysis.

We find that to check the described theory not very large simulations are needed. In Fig.1 – 6, shown above, we have used N equal to 2^{15}, i.e. 32768. No qualitative difference appears with the use of larger N or greater number of realizations.

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Figure captions:

Figure 1: A realization of the velocity field at time $t = 0$ (a), evolved process at dimensionless time $t/t_{nl} \sim 10^5$ (b), and the same realization at a larger time $t/t_{nl} = 10^6$ (c).

Figure 2: (a) Power spectral densities computed at different times: (1) $t/t_{nl} \sim 5 \cdot 10^8$, (2) $t/t_{nl} \sim 3 \cdot 10^9$, (3) $t/t_{nl} \sim 2 \cdot 10^{10}$. Log-log plot.
(b) Three power spectral densities computed at the same time. The exponent n of the initial power spectrum for the curve above $n = 3$, and for the other two, which are hardly distinguishable, $n = 6$ and $n = 12$. Log-log plots.

Figure 3: Power spectral densities resulted from evaluation of an analytical integral expression (upper curve) and from the direct numerical simulation (lower curve shifted down to be distinguishable from the upper one). Log-log plot.

Figure 4: Characteristic space-scale as a function of time. The prediction $l(t) \sim \sqrt{t}$ (dashed line) is by dimensional analysis, while the more precise approximation is $(\log t)^{-1/4}\sqrt{t}$ (solid line). Log-log plot.

Figure 5: Total energy as a function of time $E(t)$.
Centered symbols - the numerically obtained values of $E(t)$. The prediction $E(t) \sim 1/t$ (dashed line) is by dimensional analysis, while the more precise approximation is $1/t\sqrt{\log t}$ (solid line). Log-log plots.

Figure 6: The asymptotic prefactor $A(t)$ at the IR part of the power spectral density as a function of time. Centered symbols - values of $A(t)$ resulted from the numerical simulation. Dashed line - the dimensional prediction, and solid line - analytical result with taking into account the logarithmic correction.